

Principle of Relativity, Dual Poincaré Group and Relativistic Quadruple

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Abstract

Based on the principle of relativity with two universal constants (c, l) , there is the inertial motion group $IM(1, 3) \sim PGL(5, R)$. With Lorentz group $SO(1, 3)$ for isotropy, in addition to Poincaré group \mathcal{P} of Einstein's special relativity the dual Poincaré group \mathcal{P}_2 preserves the origin lightcone C_O and its space/time-like region R_\pm appeared at common origin of intersected Minkowski/de Sitter/anti-de Sitter space M/D_\pm . The \mathcal{P}_2 kinematics is on a pair of degenerate Einstein manifolds M_\pm with $\Lambda_\pm = \pm 3l^{-2}$ for R_\pm , respectively. Thus, there is a Poincaré double $[\mathcal{P}, \mathcal{P}_2]_{M/M_\pm}$. There is also the de Sitter double $[\mathcal{D}_+, \mathcal{D}_-]_{D_\pm}$ for dS/AdS special relativity. Further, related to $M/M_\pm/D_\pm$, there exist other four doubles $[\mathcal{D}_+, \mathcal{P}]_{D_+/M}$, $[\mathcal{D}_-, \mathcal{P}]_{D_-/M}$, $[\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_\pm}$, and $[\mathcal{D}_-, \mathcal{P}_2]_{D_-/M_\pm}$. These doubles form a relativistic quadruple $[\mathcal{P}, \mathcal{P}_2, \mathcal{D}_+, \mathcal{D}_-]_{M/M_\pm/D_\pm}$ for three kinds of special relativity on M/D_\pm , respectively. The dS special relativity associated with the double $[\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_+}$ should provide the consistent kinematics for cosmic scale physics of $\Lambda_+ > 0$.

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I. INTRODUCTION

Precise cosmology[1, 2] shows that our universe is accelerated expanding and is not asymptotic to the Minkowski (*Mink*) space, rather quite possibly a de Sitter (*dS*) space with a tiny positive cosmological constant $\Lambda_+ > 0$. This opens up the era of the cosmic scale physics characterized by the Λ_+ and greatly challenges Einstein's theory of relativity[3, 4] as its solid foundation. As "principle theory", Einstein's principles must be reexamined from the very beginning to explore if there is new kinematics for the cosmic scale physics. Actually, this is the case.

In order to introduce the cosmological constant at the principle level, a universal invariant constant l of length dimension should be introduced in addition to the speed of light c . This is also the case from some general and simple dimensional analysis: in space-time physics space and time coordinates should have right invariant dimension, i.e. $[x^i] = L, (i = 1, 2, 3)$ and $[t] = T$. In order to characterize them and to make coordinate t has the same dimension with x^i of dimension length, a universal constant c of velocity dimension $[c] = L/T$ and another universal constant l with $[l] = L$ should be introduced, which should also be invariant under transformations among coordinate systems. And their realistic values must be determined by experiments and observations with respect to inertial observers. In fact, c is determined as the speed of light and l is hidden in Einstein's special relativity (*SR*) and as long as the Euclidean assumption on time and space is relaxed, the principle of relativity

(PoR) and the postulate for the light propagation in SR [3] can be extended and weakened to the PoR_{cl} of two universal constants (c, l), denoted as PoR_{cl} .

Based on the PoR_{cl} , it has been found that for inertial motion described by Newton's first law there are linear fractional transformations with common denominators (LFT s) of the inertial motion group $IM(1, 3)$ of twenty-four generators homomorphic to the 4d real projective group $PGL(5, R)$ with algebra $\mathfrak{im}(1, 3) \cong \mathfrak{pgl}(5, R)$ kinematically, and it contains homogeneous Lorentz group $\mathcal{L} := SO(1, 3)$ as subgroup for relativistic cases[5, 6]. In fact, in addition to the Poincaré group \mathcal{P} with Euclid time and space translations $(H, \mathbf{P}_i) \in \mathfrak{p}$ on $Mink$ -space M , there is also the dual Poincaré group \mathcal{P}_2 of LFT s¹ with time and space pseudo-translations $(H', \mathbf{P}'_i) \in \mathfrak{p}_2 \cong \mathfrak{iso}(1, 3)$ proportional to l^{-2} that preserves the $Mink$ origin lightcone C_O and its space/time-like region R_{\pm} . So, related to M , there is an algebraic doublet $(\mathfrak{p}, \mathfrak{p}_2)$. And in terms of the linear combinations among translations

$$H \pm H' = H^{\pm}, \quad \mathbf{P}_i \pm \mathbf{P}'_i = \mathbf{P}_i^{\pm} \quad (1.1)$$

with common $\mathfrak{so}(1, 3)$ isotropy, the Poincaré doublet leads to the dS/AdS doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ with Beltrami translations $(H^{\pm}, \mathbf{P}_i^{\pm})$ in the Beltrami- dS/AdS -space D_{\pm} of radius l , respectively, and vice versa in $\mathfrak{im}(1, 3) \cong \mathfrak{pgl}(5, R)$ [5, 6]. This is very simple and different from the contraction [7, 8] and deformation[9] approaches. Thus, based on the PoR_{cl} and within $IM(1, 3) \sim PGL(5, R)$ there are three kinds of Poincaré/ dS/AdS invariant SR , respectively[10–14]. With common Lorentz isotropy and relations (1.1), they automatically form an SR triple of four related groups $\mathcal{P}, \mathcal{P}_2, \mathcal{D}_+ := SO(1, 4)$ and $\mathcal{D}_- := SO(2, 3)$ with an algebraic quadruplet $\mathfrak{q} := (\mathfrak{p}, \mathfrak{p}_2, \mathfrak{d}_+, \mathfrak{d}_-)$. It is important that these relativistic kinematics and their combinational structures automatically appear in the algebra $\mathfrak{im}(1, 3) \cong \mathfrak{pgl}(5, R)$.

In fact, by means of projective geometry method, it can be shown that for those relativistic kinematics and their algebraic relations there are also their group and geometry counterparts with common Lorentz isotropy, i.e. $\mathcal{L} = \mathcal{P} \cap \mathcal{P}_2 \cap \mathcal{D}_+ \cap \mathcal{D}_-$, with respect to the LFT s of the group $IM(1, 3) \sim PGL(5, R)$.

This can be seen from the other angle, i.e. these issues can also easily be reached by their construction starting from the $Mink$ -lightcone at the origin. Since there is the common Lorentz isotropy, as long as the flatness of $Mink$ -space is relaxed, the region defined by $\eta_{\mu\nu}x^{\mu}x^{\nu} \mp l^2 \leq 0$, i.e. shifting the origin lightcone equation by $\mp l^2$, leads to the domain of the Beltrami- dS/AdS -space D_{\pm} with x^{μ} as the Beltrami coordinates without antipodal identification, respectively. Since the origin lightcone in D_{\pm} is still Minkowskian, there are the SR triple for three kinds of SR on intersected non-degenerate $Mink/dS/AdS$ -spaces M/D_{\pm} and four related groups with the quadruplet \mathfrak{q} [5, 6]. As the \mathcal{P}_2 -kinematics, there is a pair of \mathcal{P}_2 invariant degenerate Einstein manifolds² $M_{\pm} := (M_{\pm}^{\mathcal{P}_2}, \mathbf{g}_{\pm}, \mathbf{g}_{\pm}^{-1}, \nabla_{\Gamma_{\pm}})$ with cosmological constant $\Lambda_{\pm} = \pm 3/l^2$, where the degenerate metric \mathbf{g}_{\pm} is induced from the C_O as an absolute for its space/time-like region R_{\pm} on $Mink$ -space M , respectively, with its formal inverse \mathbf{g}_{\pm}^{-1} and the Christoffel symbol Γ_{\pm} . Then for four groups $\mathcal{P}/\mathcal{P}_2/\mathcal{D}_{\pm}$ as subgroup of $IM(1, 3)$, corresponding to their algebraic relations (1.1) they should share the Lorentz group \mathcal{L}

¹ It is also called the second Poincaré group in [5, 6].

² Einstein manifold means that it satisfies vacuum Einstein field equation only.

for isotropy, and on these spaces $M/M_{\pm}/D_{\pm}$ there is the same origin lightcone structure $[C_O]$ in the sense of $R_{\pm} = M \cap M_{\pm} \cap D_+ \cap D_-$. So, the \mathcal{P}_2 -kinematics for R_{\pm} induced from $C_O = \partial R_{\pm}$ is also common and with common Lorentz isotropy there are six types of doubles, i.e. $[\mathcal{P}, \mathcal{P}_2]_{M/M_{\pm}}$, $[\mathcal{D}_+, \mathcal{P}]_{D_+/M}$, $[\mathcal{D}_-, \mathcal{P}]_{D_-/M}$, $[\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_{\pm}}$, $[\mathcal{D}_-, \mathcal{P}_2]_{D_-/M_{\pm}}$, and $[\mathcal{D}_+, \mathcal{D}_-]_{D_{\pm}}$. Thus, based on the PoR_{cl} , they automatically form a relativistic quadruple $\mathfrak{Q}_{PoR} := [\mathcal{P}, \mathcal{P}_2, \mathcal{D}_+, \mathcal{D}_-]_{M/M_{\pm}/D_{\pm}}$.

Since there is the dual Poincaré group \mathcal{P}_2 and its kinematics for R_{\pm} at origin, related to *Mink*-space M , the symmetry and geometry are dramatically changed by the Poincaré double $[\mathcal{P}, \mathcal{P}_2]_{M/M_{\pm}}$. In order to restore Einstein's *SR*, $l = \infty$ has to be taken. But, for the relativistic physics as a description of reality on free space without gravity in the universe, the cosmological constant Λ_+ cannot be missed. Then the (H', \mathbf{P}'_i) can be ignored up to

$$\nu^2 := (c/l)^2 \simeq c^2 \Lambda_+ / 3 \sim 10^{-35} sec^{-2}, \quad (1.2)$$

where $\nu := c/l$ is called the Newton-Hooke constant. Thus, their effects can still be ignored up to now locally except for the cosmic scale physics.

The paper is arranged as follows. In sec. II, we briefly recall the inertial motion group based on the PoR_{cl} and the relativistic kinematics that with common Lorentz isotropy act naturally as the *SR* triple with four related algebras. In sec. III, we show that there is the dual Poincaré group \mathcal{P}_2 and its kinematics on a pair of degenerate Einstein manifolds M_{\pm} induced from the origin lightcone C_O as absolute for its space/time-like region R_{\pm} , respectively. Thus, there is the Poincaré double $[\mathcal{P}, \mathcal{P}_2]_{M/M_{\pm}}$ consists of the Poincaré group \mathcal{P} on *Mink*-space and the \mathcal{P}_2 -kinematics on M_{\pm} for R_{\pm} , respectively. In sec. IV, we show that with common Lorentz isotropy not only the *SR* on Beltrami-*dS*/*AdS* spaces are related, but their domain conditions and absolutes at origin also become the R_{\pm} . This can also be reached from the shifted *Mink*-lightcone by relaxing the flatness of *Mink*-space. Hence, there are naturally six doubles that form the quadruple \mathfrak{Q}_{PoR} for the *SR* triple. Finally, we end with some remarks.

II. PRINCIPLE OF RELATIVITY WITH TWO UNIVERSAL CONSTANT AND INERTIAL MOTION GROUP

As was just emphasized, in order to explore whether there is new kinematics for the cosmic scale physics, the *PoR* and the postulate for the light propagation in Einstein's *SR* [3] should be extended and weakened to the *PoR* of two universal constants (c, l) , i.e. the PoR_{cl} . Let us consider kinematic aspects of the PoR_{cl} . That is what should be the symmetry for the inertial motions with respect to the PoR_{cl} . It turns out there is a group for inertial motions described by Newton's law of inertia called the inertial motion group, which is homomorphic to 4d real projective group $PGL(5, R)$, and all kinematics with space isotropy of ten generators with some missed before.

A. Inertial Motion Transformation Group for Newton's Law of Inertia

Umov[15], Weyl[16], Fock[17] and Hua[18, 19] studied an important issue: What are the most general transformations among the inertial frames $\mathcal{F} := \{S(x)\}$ that keep the inertial motions?

In view the PoR_{cl} , the answer is: *The most general transformations among \mathcal{F} that keep inertial motion described by Newton's first law*

$$x^i = x_0^i + v^i(t - t_0), \quad v^i = \frac{dx^i}{dt} = \text{const.} \quad i = 1, 2, 3, \quad (2.1)$$

are the *LFTs of twenty-four parameters*

$$\mathbf{T} : \quad l^{-1}x'^\mu = \frac{A^\mu_\nu l^{-1}x^\nu + b^\mu}{c_\lambda l^{-1}x^\lambda + d}, \quad x^0 = ct, \quad (2.2)$$

$$\det \mathbf{T} = \begin{vmatrix} A & b^t \\ c & d \end{vmatrix} \neq 0, \quad (2.3)$$

where $A = (A^\mu_\nu)$ a 4×4 matrix, b, c 1×4 matrixes, $d \in R$, $c_\lambda x^\lambda = \eta_{\lambda\sigma} c^\lambda x^\sigma$, t for transpose and $J = (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. These LFTs form the inertial motion group $IM(1, 3)$ homomorphic to the 4d real projective group $PGL(5, R)$, i.e. $\forall \mathbf{T} \in IM(1, 3) \sim PGL(5, R)$, with the inertial motion algebra $\mathfrak{im}(1, 3)$ isomorphic to $\mathfrak{pgl}(5, R)$. Further, the time reversal \mathbf{T} and space inversion \mathbf{P} preserve the inertial motion (2.1). So, all issues are modulo the \mathbf{T} and \mathbf{P} invariance.

In fact, inertial motion (2.1) can be viewed as a 4d straightline [18, 19]. In projective geometry, LFTs (2.2) form the real projective group $PGL(5, R)$ [19] with algebra $\mathfrak{pgl}(5, R)$. But, for orientation in physics the antipodal identification of 4d real projective space should not be taken so that $IM(1, 3) \sim PGL(5, R)$. Hereafter, we still call it the projective geometry approach in this sense.

B. Inertial Motion Algebra and Relativistic Kinematic Algebras

It is straightforward to get the generator set $\{T\}^{\text{im}} := (H^\pm, \mathbf{P}_i^\pm, \mathbf{J}_i, \mathbf{K}_i, \mathbf{N}_i, R_{ij}, M_\mu)$ of LFTs (2.2) where

$$\begin{aligned} \mathbf{N}_i &:= t\partial_i + c^{-2}x_i\partial_t, \\ R_{ij} &:= x_i\partial_j - x_j\partial_i, (i < j), \quad M_\mu := x^{(\mu)}\partial_{(\mu)}, \end{aligned} \quad (2.4)$$

where no summation for repeated indexes in brackets. It is important that either $\mathfrak{so}(1, 3)$ of Lorentz group \mathcal{L} generated by space rotation \mathbf{J}_i defined as

$$\mathbf{J}_i = \frac{1}{2}\epsilon_i^{jk}L_{jk}, \quad L_{jk} := x_j\partial_k - x_k\partial_j, \quad (2.5)$$

and Lorentz boosts \mathbf{K}_i defined as

$$\mathbf{K}_i := t\partial_i - c^{-2}x_i\partial_t, \quad (2.6)$$

or $\mathfrak{so}(3)$ of its subgroup $SO(3)$ generated by space rotation \mathbf{J}_i , both groups are subset of $A^\mu_\nu \subset T$ in (2.2). So the Lorentz isotropy for relativistic cases or space isotropy for all cases are common for relevant kinematics, respectively. Among four time and space translations $\{\mathcal{H}\} := \{H, H', H^\pm\}$

of dimension $[\nu]$, $\{\mathbf{P}\} := \{\mathbf{P}_i, \mathbf{P}'_i, \mathbf{P}_i^\pm\}$ of dimension $[l^{-1}]$, and four boosts $\{\mathbf{K}\} := \{\mathbf{K}_i, \mathbf{N}_i, \mathbf{K}_i^g, \mathbf{K}_i^c\}$ of Lorentz, geometry, Galilei and Carroll boosts of dimension $[c^{-1}]$, there are two independents, respectively

$$H := \partial_t, \quad H' := -\nu^2 t x^\nu \partial_\nu, \quad H^\pm := \partial_t \mp \nu^2 t x^\nu \partial_\nu; \quad (2.7)$$

$$\mathbf{P}_i := \partial_i, \quad \mathbf{P}'_i := -l^{-2} x_i x^\nu \partial_\nu, \quad \mathbf{P}_i^\pm := \partial_i \mp l^{-2} x_i x^\nu \partial_\nu; \quad (2.8)$$

$$\begin{aligned} \mathbf{K}_i &:= t \partial_i - c^{-2} x_i \partial_t, \quad \mathbf{N}_i := t \partial_i + c^{-2} x_i \partial_t, \\ \mathbf{K}_i^g &:= t \partial_i, \quad \mathbf{K}_i^c := -c^{-2} x_i \partial_t. \end{aligned} \quad (2.9)$$

These generators are scalar and vector representation of $\mathfrak{so}(3)$ generated by \mathbf{J}_i without dimension, as follows[5, 6]

$$[\mathbf{J}, \mathbf{J}] = \mathbf{J}, \quad [\mathbf{J}, \mathcal{H}] = 0, \quad [\mathbf{J}, \mathbf{P}] = \mathbf{P}, \quad [\mathbf{J}, \mathbf{K}] = \mathbf{K}, \quad (2.10)$$

where with $\epsilon_{123} = -\epsilon_{12}^3 = 1$ $[\mathbf{J}, \mathbf{P}] = \mathbf{P}$ is, e.g. a shorthand of $[\mathbf{J}_i, \mathbf{P}_j^\pm] = -\epsilon_{ij}^k \mathbf{P}_k^\pm$ etc. All generators and commutators have right dimensions expressed by the constants c, l or ν . In addition to combination (1.1), there are also combinatory relations $\mathbf{K}_j / \mathbf{N}_j = \mathbf{K}_i^g \pm \mathbf{K}_i^c$ for boosts.

Then the generator set $\{T\}^{\text{im}}$ spans the $\mathfrak{im}(1, 3)$ as follows [6],

$$\begin{aligned} [\mathbf{P}_i^+, \mathbf{P}_j^-] &= (1 - \delta_{(i)(j)}) l^{-2} R_{(i)(j)} - 2l^{-2} \delta_{i(j)} (M_{(j)} + D), \\ [\mathbf{P}_i^\pm, M_j] &= \delta_{i(j)} \mathbf{P}_{(j)}^\mp, & [\mathbf{P}_i^\pm, R_{jk}] &= -\delta_{ij} \mathbf{P}_k^\mp - \delta_{ik} \mathbf{P}_j^\mp, \\ [H^\pm, H^-] &= 2\nu^2 (M_0 + D), & [H^\pm, M_0] &= H^\mp, \\ [\mathbf{K}_i, M_0] &= -\mathbf{N}_i, & [\mathbf{K}_i, M_j] &= \delta_{i(j)} \mathbf{N}_{(j)}, \\ [\mathbf{K}_i, R_{jk}] &= -\delta_{ij} \mathbf{N}_k - \delta_{ik} \mathbf{N}_j, & [\mathbf{N}_i, M_0] &= -\mathbf{K}_i, \\ [\mathbf{N}_i, M_j] &= \delta_{i(j)} \mathbf{K}_{(j)}, & [\mathbf{N}_i, R_{jk}] &= -\delta_{ij} \mathbf{K}_k - \delta_{ik} \mathbf{K}_j, \\ [\mathbf{K}_i, \mathbf{N}_j] &= (\delta_{(i)(j)} - 1) c^{-2} R_{(i)(j)} - 2\delta_{(i)j} c^{-2} (M_0 - M_{(i)}), \\ [L_{ij}, M_k] &= \delta_{j(k)} R_{i(k)} - \delta_{i(k)} R_{j(k)}, & [R_{ij}, M_k] &= \delta_{i(k)} L_{j(k)} + \delta_{j(k)} L_{i(k)}, \\ [L_{ij}, R_{kl}] &= 2(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) (M_i - M_j) + \delta_{ik} R_{jl} + \delta_{il} R_{jk} - \delta_{jk} R_{il} - \delta_{jl} R_{ik}, \\ [R_{ij}, R_{kl}] &= -\delta_{ik} L_{jl} - \delta_{il} L_{jk} - \delta_{jk} L_{il} - \delta_{jl} L_{ik}, \end{aligned} \quad (2.11)$$

where D is called the dilation generator and is defined as

$$D := \sum_{\kappa} M_{\kappa}. \quad (2.12)$$

In Table I, all relativistic kinematics are listed symbolically. For the geometrical and non-relativistic cases, we shall study them in detail elsewhere.

TABLE I: All relativistic kinematics

Group	Algebra	Generator set	$[\mathcal{H}, \mathbf{P}]$	$[\mathcal{H}, \mathbf{K}]$	$[\mathbf{P}, \mathbf{P}]$	$[\mathbf{K}, \mathbf{K}]$	$[\mathbf{P}, \mathbf{K}]$
\mathcal{P}	\mathfrak{p}	$(H, \mathbf{P}_i, \mathbf{K}_i, \mathbf{J}_i)$	0	\mathbf{P}	0	$-c^{-2} \mathbf{J}$	$c^{-2} \mathcal{H}$
\mathcal{P}_2	\mathfrak{p}_2	$\{H', \mathbf{P}'_i, \mathbf{K}_i, \mathbf{J}_i\}$					
\mathcal{D}_+	\mathfrak{d}_+	$(H^+, \mathbf{P}_i^+, \mathbf{K}_i, \mathbf{J}_i)$	$\nu^2 \mathbf{K}$	\mathbf{P}	$l^{-2} \mathbf{J}$	$-c^{-2} \mathbf{J}$	$c^{-2} \mathcal{H}$
\mathcal{D}_-	\mathfrak{d}_-	$(H^-, \mathbf{P}_i^-, \mathbf{K}_i, \mathbf{J}_i)$	$-\nu^2 \mathbf{K}$	\mathbf{P}	$-l^{-2} \mathbf{J}$	$-c^{-2} \mathbf{J}$	$c^{-2} \mathcal{H}$

It is clear that with common Lorentz isotropy $\mathfrak{so}(1, 3)$ spanned by $(\mathbf{K}_i, \mathbf{J}_i)$, the Poincaré doublet $(\mathfrak{p}, \mathfrak{p}_2)$ leads to the dS/AdS doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ with the Beltrami time and space translations $(H^\pm, \mathbf{P}_j^\pm)$ in the Beltrami- dS/AdS -space, respectively, and vice versa[5, 6]. In addition, from the algebraic relations of $\mathfrak{im}(1, 3)$ [6] both $(\mathfrak{p}, \mathfrak{p}_2)$ and $(\mathfrak{d}_+, \mathfrak{d}_-)$ are closed in the $\mathfrak{im}(1, 3)$, while the generators (R_{ij}, M_μ) , which generate A^μ_ν in LFT s (2.2) together with \mathbf{N}_i in (2.9) and $(\mathbf{K}_i, \mathbf{J}_i)$, exchange the translations from one to another in $(\mathfrak{p}, \mathfrak{p}_2)$ and $(\mathfrak{d}_+, \mathfrak{d}_-)$. This has been shown in [5, 6]. Since all issues are in $IM(1, 3)$ based on the PoR_{cl} , there should be three kinds of SR [10–14] with Newton’s law of inertia that automatically form the SR triple of four related algebras $\mathfrak{so}(1, 3) = \mathfrak{p} \cap \mathfrak{p}_2 \cap \mathfrak{d}_+ \cap \mathfrak{d}_-$ that form the algebraic quadruplet \mathfrak{q} [6].

We will show that this also the case for their relevant groups and geometries.

III. DUAL POINCARÉ GROUP, \mathcal{P}_2 -KINEMATICS AND POINCARÉ DOUBLE

Let us now consider why together with usual Poincaré group there is a dual Poincaré group and its role in relativistic physics.

A. Poincaré Group and Dual Poincaré Group

Under usual Poincaré group $\mathcal{P} := ISO(1, 3) = R(1, 3) \ltimes \mathcal{L}$

$$P : x'^\mu = (x^\nu - a^\mu)L^\mu_\nu, \det P = \det \begin{pmatrix} L & b^t \\ 0 & 1 \end{pmatrix} \neq 0, \quad (3.1)$$

where $L = (L^\mu_\nu) \in SO(1, 3)$, $b^t = -l^{-1}(aL)^t$, and $\forall P \in \mathcal{P}$. Then, it follows the *Mink*-space $M := \mathcal{P}/\mathcal{L}$ with the *Mink*-metric and the *Mink*-lightcone at event $A(a^\mu)$

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu = dx J dx^t, \\ C_A : (x - a)J(x - a)^t &= 0, \end{aligned} \quad (3.2)$$

with the lightcone structure $[C_A]$:

$$[C_A] : (x - a)J(x - a)^t \gtrless 0, \quad ds^2|_A = dx J dx^t|_A = 0. \quad (3.3)$$

The generator set $\{T\}^\mathfrak{p} := (H, \mathbf{P}_i, \mathbf{K}_i, \mathbf{J}_i)$ spans $\mathfrak{p} := \mathfrak{iso}(1, 3)$ listed in Tale I.

It is important that based on the PoR_{cl} , in addition to this usual Poincaré group \mathcal{P} , there is another group isomorphic to it called the dual Poincaré group $\mathcal{P}_2 \cong ISO(1, 3)$. Let us now consider this group and its kinematics.

In fact, the symmetry of the origin lightcone C_O with its space/time-like region R_\pm

$$C_O : \quad x J x^t = 0 \quad (3.4)$$

$$R_\pm : \quad r_\pm(x) := \mp x J x^t > 0 \quad (3.5)$$

is not just the Lorentz group, but the semi-product of the dual Poincaré group \mathcal{P}_2 and a dilation generated by (2.12).

By means of the projective geometry method, C_O and R_\pm should be regarded as the absolute and the domain, respectively. Then, LFT s (2.2) reduce to their subset:

$$l^{-1}x^\mu \rightarrow l^{-1}x'^\mu = \frac{L^\mu{}_\nu l^{-1}x^\nu}{c_\lambda l^{-1}x^\lambda + d}, \quad (3.6)$$

in which the LFT s for $d = 1$ form the dual Poincaré group, $\forall P_2 \in \mathcal{P}_2 \cong ISO(1, 3)$,

$$\det P_2 = \det \begin{pmatrix} L & 0 \\ c & d \end{pmatrix} \neq 0, \quad \text{for } d = 1. \quad (3.7)$$

This can be easily seen from that each matrix P_2 is isomorphic to the transport of a matrix P in (3.1). It is also easy to show that this group preserves C_O and R_\pm so does $[C_O]$ in (3.3) with $A(a^\mu) = O(0^\mu)$. The generator of dilation d in (3.6) is D (2.12).

The generator set $\{T\}^{\mathfrak{p}_2} = (H', \mathbf{P}'_i, \mathbf{K}_i, \mathbf{J}_i)$ of \mathcal{P}_2 consists of

$$\begin{aligned} H' &:= -\nu^2 t x^\nu \partial_\nu, & \mathbf{P}'_i &:= -l^{-2} x_i x^\nu \partial_\nu, \\ \mathbf{K}_i &:= t \partial_i - c^{-2} x_i \partial_t, & \mathbf{J}_i &:= \frac{1}{2} \epsilon_i^{jk} L_{jk}, \end{aligned} \quad (3.8)$$

spans $\mathfrak{p}_2 \cong \mathfrak{iso}(1, 3)$ for $[C_O]$ listed in Table I. The generator D for the dilation also keeps $[C_O]$.

In fact, the dual Poincaré group is also based on the PoR . However, it had been simply reduced to the homogeneous Lorentz group in [3], in which Einstein wrote:

At the time $t = \tau = 0$, when the origin of the co-ordinates is common to the two frames, let a spherical wave be emitted therefrom, and be propagated with the velocity c in system K . If (x, y, z) be a point just attained by this wave, then

$$x^2 + y^2 + z^2 = c^2 t^2.$$

Transforming this equation with the aid of our equations of transformation we obtain after a simple calculation

$$\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2.$$

The wave under consideration is therefore no less a spherical wave with velocity of propagation c when viewed in the moving system. This shows that our two fundamental principles are compatible^a.

^a The equations of the Lorentz transformation may be more simply deduced directly from the condition that in virtue of those equations the relation $x^2 + y^2 + z^2 = c^2 t^2$ shall have as its consequence the second relation $\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2$.

In the above footnote by Einstein, H' and \mathbf{P}'_i had been missed.

As is well-known, for the light cone, the maximum invariant symmetry is conformal group $SO(2, 4)$. However, its special conformal transformations do not preserve the inertial motion (2.1). It should also be mentioned that for $L^\mu{}_\nu = \delta^\mu{}_\nu$, transformations (3.6) look like “local scale transformations” at first glance. In fact, this is not true. The local scale transformations should not include other group parameters c_λ and d , which do not depend on x .

B. Dual Poincaré Group as Kinematics on Degenerate Einstein Manifolds

As a kinematic symmetry, what kind of spacetimes should be transformed under the dual Poincaré group? Namely, what about the dual Poincaré group \mathcal{P}_2 invariant kinematics?

It can be show that the \mathcal{P}_2 invariant 4d metric must be degenerate[20, 21]. We will study this issue together others by means of the projective geometry method as well.

For a pair of events $A(a^\mu), B(b^\mu) \in R_\pm$, a line between them

$$L : (1 - \tau)a + \tau b \quad (3.9)$$

crosses $C_O := \partial(R_\pm)$ at τ_1 and τ_2 , which satisfy

$$(b - a)J(b - a)^t \tau^2 + 2aJ(b - a)^t + (aJa^t \mp l^2) = 0. \quad (3.10)$$

For four events with $\tau = (\tau_1, 1, \tau_2, 0)$, a cross ratio can be given. From the power 2 cross ratio invariant

$$\begin{aligned} \Delta_{R_\pm}^2(A, X) &= \pm l^2 \{r_\pm^{-1}(a)r_\pm^{-1}(x)r_\pm^2(a, x) - 1\}, \\ r_\pm(a) &:= r_\pm(a, a) > 0, \quad r_\pm(a, x) := \mp aJx^t, \end{aligned} \quad (3.11)$$

for $X(x^\mu), X + dX(x^\mu + dx^\mu) \in R_\pm$, it follows a degenerate metric

$$\begin{aligned} g_{\pm\mu\nu} &= \left(\frac{\eta_{\mu\nu}}{r_\pm(x)} \pm l^{-2} \frac{x_\mu x_\nu}{r_\pm^2(x)}\right), \quad x_\mu = \eta_{\mu\lambda}x^\lambda, \\ \mathbf{g}_\pm &:= (g_\pm)_{\mu\nu} = r_\pm^{-1}(x)(J \pm l^{-2} \frac{Jx^t x J}{r_\pm(x)}). \end{aligned} \quad (3.12)$$

Although metric (3.12) is degenerate, i.e. $\det \mathbf{g}_\pm = 0$, formally there is still a contra-variant metric as its inverse, respectively:

$$\mathbf{g}_\pm^{-1} = r_\pm(x)(J \pm l^{-2} \frac{Jx^t x J}{r_\pm(x)})^{-1}, \quad (3.13)$$

which is divergent. But, their Christoffel symbol can still be formally calculated. This meaningfully results

$$\Gamma_{\pm\mu\nu}^\lambda(x) = \pm r_\pm^{-1}(x)(\delta_\mu^\lambda x_\nu + \delta_\nu^\lambda x_\mu), \quad (3.14)$$

which is obviously metric compatible, i.e.

$$\nabla_{\Gamma_\pm} \mathbf{g}_\pm = 0, \quad \nabla_{\Gamma_\pm} \mathbf{g}_\pm^{-1} = 0. \quad (3.15)$$

And it is straightforward to get the Riemann, Ricci and scale curvature, respectively, as follows

$$R_{\pm\nu\lambda\sigma}^\mu(x) = \pm l^{-2}(g_{\pm\nu\lambda}\delta_\sigma^\mu - g_{\pm\nu\sigma}\delta_\lambda^\mu) \quad (3.16)$$

$$R_{\pm\mu\nu}(x) = \pm 3l^{-2}g_{\mu\nu}^\pm, \quad R_\pm(x) = \pm 12l^{-2}. \quad (3.17)$$

Then the $M_\pm := (M_\pm^{\mathcal{P}_2}, \mathbf{g}_\pm, \mathbf{g}_\pm^{-1}, \nabla_{\Gamma_\pm})$ is an Einstein manifold with $\Lambda_\pm = \pm 3/l^2$ for R_\pm (3.5), respectively. It easy to check that the Lie derivatives of these objects vanish with respect to the

\mathfrak{p}_2 -generators as Killing vectors on M_\pm , respectively. And Eq. (2.1) is indeed the equation of motion for a free particle, if any, on such a pair of Einstein manifolds M_\pm . Namely, there is also Newton's first law of inertia.

Since all these objects are given originally from the cross ratio invariance of C_O (3.4) under LFT s (2.2) based on the PoR_{cl} , such a \mathcal{P}_2 -kinematics is invariant under (3.7). Another \mathcal{P}_2 -degenerate geometry for R_+ is given in a different manner [21].

C. The Poincaré Double

In fact, the dual Poincaré group \mathcal{P}_2 also exists for lightcone structure $[C_A]$ (3.3) while the \mathcal{P}_2 for $[C_O]$ (3.4) is a representative among them. Then there are intersections for \mathcal{P} and \mathcal{P}_2 as well as for M and M_\pm , i.e. $\mathcal{L} = \mathcal{P} \cap \mathcal{P}_2$ and $R_\pm = M \cap M_\pm$, respectively. Thus, related to the *Mink*-space there exist infinite many Poincaré doubles with the Poincaré double $[\mathcal{P}, \mathcal{P}_2]_{M/M_\pm}$ at the origin as a representative for M and a pair of M_\pm induced from C_O (3.4) for R_\pm (3.5), respectively.

Actually, under transformations (3.1) of \mathcal{P} , the Poincaré algebraic doublet $(\mathfrak{p}, \mathfrak{p}_2)$ at origin $O(o^\mu)$ can be transformed to $A(a^\mu)$ and vice versa. In fact, under \mathcal{P} , the generators $\mathbf{P}'_\mu := (H', \mathbf{P}'_i)$ as a 4-vector on M are transferred and the action is closed in the algebra $\mathfrak{im}(1, 3)$

$$\mathcal{L}_{\mathbf{P}_\mu} \mathbf{P}'_\nu = [\mathbf{P}_\mu, \mathbf{P}'_\nu] \in \mathfrak{im}(1, 3) \cong \mathfrak{pgl}(5, R), \quad (3.18)$$

in which there are dS/AdS algebras \mathfrak{d}_\pm for dS/AdS SR as subalgebras.

It should be emphasized that the existence of the dual Poincaré symmetry is indicated by a theorem in projective geometry: In the group $PGL(2, K)$ of 1d projective space over field K , the invariant group of any given point is isomorphic to the affine subgroup[22]³. This theorem can also be extended to $PGL(5, R)$ of 4d real projective space. If the Euclidian metric $\delta_{\mu\nu}$ related to the 4d affine space is changed to $\eta_{\mu\nu}$, the fixed point is then changed to the *Mink* lightcone and the dual Poincaré group appears.

Thus, symmetry and geometry related to the *Mink*-space are dramatically changed. In order to eliminate the double and restore Einstein's SR , $l = \infty$ should be taken. But, the cosmological constant Λ_+ leads to $[\mathcal{P}, \mathcal{P}_2]_{M/M_+}$ with bound (1.2) for modern relativistic physics.

IV. RELATIVISTIC QUADRUPLE WITH COMMON LORENTZ ISOTROPY

As was shown in [5, 6], dual to the Poincaré algebraic doublet $(\mathfrak{p}, \mathfrak{p}_2)$ at the origin, there is a dS/AdS algebraic doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ at the same origin. And together other four doublets at the origin, there is a relativistic algebraic quadruplet $\mathfrak{q} = (\mathfrak{p}, \mathfrak{p}_2, \mathfrak{d}_+, \mathfrak{d}_-)$ at the origin, which is in fact a

³ The PR^1 is a 1d compact and differentiable manifold with its transformation group $PGL(2, R)$. There are at least two inhomogeneous coordinate patches to cover it. In each of them LFT s (2.2) may transform a point to its infinite, which must be included so the patch is extended to PR^1 . And in each of such extended patches the theorem holds. In the intersection, the affine subgroups in one extended patch is just the dual ones in the other.

representative of infinite many such kind of quadruplets. We will show this is also the case for group and geometry aspects.

A. De Sitter, Anti-de Sitter Relativistic Kinematics and De Sitter Double

First, corresponding to the algebraic doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ with common Lorentz isotropy algebra $\mathfrak{so}(1, 3)$, there is the dS/AdS double $[\mathcal{D}_+, \mathcal{D}_-]_{D_\pm}$ of common Lorentz isotropy group $\mathcal{L} \subset IM(1, 3)$ at the origin.

In fact, in terms of homogeneous projective coordinates the dS/AdS -hyperboloid \mathcal{H}_\pm and their boundaries can be expressed, respectively

$$\mathcal{H}_\pm : \quad \eta_{\mu\nu} \xi^\mu \xi^\nu \mp (\xi^4)^2 \leq 0, \quad (4.1)$$

$$\mathcal{B}_\pm = \partial\mathcal{H}_\pm : \quad \eta_{\mu\nu} \xi^\mu \xi^\nu \mp (\xi^4)^2 = 0. \quad (4.2)$$

It is clear that they are invariant under dS/AdS group, i.e. $\mathcal{D}_\pm := SO(1, 4)/SO(2, 3)$, and the $\eta_{\mu\nu} \xi^\mu \xi^\nu$ and $(\xi^4)^2$ are in intersections of the \mathcal{H}_\pm and their boundaries $\mathcal{B}_\pm = \partial\mathcal{H}_\pm$, respectively. These imply that they just share the common Lorentz isotropy as required.

In terms of the Beltrami coordinates as the inhomogeneous projective coordinates without antipodal identification, i.e. x^μ in a chart U_4 , say,

$$x^\mu = l\xi^{4^{-1}}\xi^\mu, \quad \xi^4 > 0, \quad (4.3)$$

the dS/AdS -hyperboloid \mathcal{H}_\pm (4.1) and their boundaries (4.2) become the domain conditions and absolutes for the Beltrami- dS/AdS -space of radius l with common Lorentz isotropy, respectively

$$\mathfrak{D}_\pm : \quad \sigma_\pm(x) := \sigma_\pm(x, x) = 1 \mp l^{-2} x J x^t > 0, \quad (4.4)$$

$$\mathfrak{B}_\pm : \quad \sigma_\pm(x) = 1 \mp l^{-2} x J x^t = 0. \quad (4.5)$$

Then, by means of the projective geometry method, the intersected Beltrami- dS/AdS -spaces D_\pm of \mathcal{D}_\pm invariant can be set up and form the dS/AdS double $[\mathcal{D}_+, \mathcal{D}_-]_{D_\pm}$ with common Lorentz isotropy, i.e. $\mathcal{L} = \mathcal{D}_+ \cap \mathcal{D}_-$.

In fact, LFT s (2.2) with (4.5) as absolute are reduced to the dS/AdS - LFT s with common $L^\mu_\kappa \in \mathcal{L}$

$$\begin{aligned} \mathcal{D}_\pm : \quad x^\mu &\rightarrow x'^\mu = \pm \sigma_\pm^{1/2}(a) \sigma_\pm^{-1}(a, x) (x^\nu - a^\nu) D_\pm^\mu{}_\nu, \\ D_\pm^\mu{}_\nu &= L^\mu{}_\nu \pm l^{-2} a_\nu a^\kappa (\sigma_\pm(a) + \sigma_\pm^{1/2}(a))^{-1} L^\mu{}_\kappa. \end{aligned} \quad (4.6)$$

This is the same as given before in [10, 11, 13]. As the LFT s of \mathcal{D}_\pm , (4.6) preserves (4.4) for the Beltrami- dS/AdS space D_\pm , respectively.

For a pair of events $A(a^\mu), B(b^\mu) \in \mathfrak{D}_\pm$, a line (3.9) between them crosses the absolutes \mathfrak{B}_\pm at τ_1, τ_2 . For four events with $\tau = (\tau_1, 1, \tau_2, 0)$, a cross ratio can be given. Further, from a power 2 cross ratio invariant the following interval between a pair of events $A(a^\mu)$ and $X(x^\mu)$ and the lightcone with top at $A(a^\mu)$ follows, respectively

$$\Delta_\pm^2(A, X) = \pm l^{-2} \{ \sigma_\pm^{-1}(a) \sigma_\pm^{-1}(x) \sigma_\pm^2(a, x) - 1 \} \gtrless 0, \quad (4.7)$$

$$\mathcal{F}_\pm : \quad \sigma_\pm^2(a, x) - \sigma_\pm(a) \sigma_\pm(x) = 0. \quad (4.8)$$

For the closely nearby two events $X(x^\mu), X + dX(x^\mu + dx^\mu) \in \mathfrak{D}_\pm$, the Beltrami metric[10, 11, 13] follows from (4.7)

$$ds_\pm^2 = \left(\frac{\eta_{\mu\nu}}{\sigma_\pm(x)} \pm l^{-2} \frac{x_\mu x_\nu}{\sigma_\pm^2(x)} \right) dx^\mu dx^\nu, \quad \sigma_\pm(x) > 0. \quad (4.9)$$

That is

$$\mathbf{g}_\pm^B := (g_\pm^B)_{\mu\nu} = \sigma_\pm^{-1}(x) \left(J \pm l^{-2} \frac{J x^t x J}{\sigma_\pm(x)} \right). \quad (4.10)$$

Its inverse as the contravariant metric reads

$$\mathbf{g}_\pm^{B-1} = \sigma_\pm(x) \left(J \pm l^{-2} \frac{J x^t x J}{\sigma_\pm(x)} \right)^{-1}. \quad (4.11)$$

Due to transitivity of (4.6), the Beltrami- dS/AdS space $D_\pm \cong \mathcal{D}_\pm/\mathcal{L}$ with $[C_O] = D_+ \cap D_-$ is homogeneous, respectively. It is also true for the entire Beltrami- dS/AdS space globally. And the generator sets $\{T^{\mathfrak{D}\pm}\} = (H^\pm, \mathbf{P}_i^\pm, \mathbf{K}_i, \mathbf{J}_i)$ of (4.6), i.e.

$$\begin{aligned} H^\pm &:= \partial_t \mp \nu^2 t x^\nu \partial_\nu, \quad \mathbf{P}_i^\pm := \partial_i \mp l^{-2} x_i x^\nu \partial_\nu, \\ \mathbf{K}_i &:= t \partial_i - c^{-2} x_i \partial_t, \quad \mathbf{J}_i = \frac{1}{2} \epsilon_i^{jk} L_{jk}, \end{aligned} \quad (4.12)$$

span the dS/AdS -doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ listed in Table I, respectively.

It is straightforward to calculate its Christoffel connection for the Beltrami- dS/AdS space. Namely,

$$\Gamma_{\pm\mu\nu}^{B\lambda}(x) = \pm l^{-2} \sigma_\pm^{-1}(x) (\delta_\mu^\lambda x_\nu + \delta_\nu^\lambda x_\mu), \quad (4.13)$$

which is obviously metric compatible. And it is straightforward to get the Riemann, Ricci and scale curvature, respectively, as follows

$$\begin{aligned} R_{\pm\nu\lambda\sigma}^{B\mu}(x) &= \pm l^{-2} (g_{\pm\nu\lambda}^B \delta_\sigma^\mu - g_{\pm\nu\sigma}^B \delta_\lambda^\mu) \\ R_{\pm\mu\nu}^B(x) &= \pm 3l^{-2} g_{\pm\mu\nu}^B, \quad R_\pm^B(x) = \pm 12l^{-2}. \end{aligned} \quad (4.14)$$

Then it is the positive/negative constant curvature Einstein manifold with $\Lambda_\pm = \pm 3l^{-2}$, respectively.

In addition, the generators in the sets $\{T^{\mathfrak{D}\pm}\} = (H^\pm, \mathbf{P}_i^\pm, \mathbf{K}_i, \mathbf{J}_i)$ of the dS/AdS algebra \mathfrak{d}_\pm can be regarded as Killing vectors of Beltrami- dS/AdS metric (4.9). And with respect to these Killing vectors the Lie derivatives vanish for the metric, connection and curvature.

Further, it is straightforward to check that the geodesic motion of metric (4.9) is indeed the inertial motion (2.1) of Newton's law of inertia as was shown in [10, 11, 13].

It is interesting to see that if origin lightcone equation (3.4) is shifted by $\mp l^2$, Eqs (4.4) can be reached as a pair of related regions on *Mink*-space M with boundaries (4.5) as a 'pseudosphere', respectively. This has been done long ago by Minkowski and others (see, e.g. [23]). If the flatness of space is relaxed, they are just the domain conditions and absolutes for the Beltrami- dS/AdS spaces D_\pm of dS/AdS -invariance. They indeed share the same origin lightcone structure $[C_O]$ from (4.8) and (4.9), i.e. $[C_O] = D_+ \cap D_-$ and $\mathcal{L} = \mathcal{D}_+ \cap \mathcal{D}_-$ at common origin. So, the dS/AdS SR on D_\pm form a double $[\mathcal{D}_+, \mathcal{D}_-]_{D_\pm}$.

B. Special Relativity Triple and Relativistic Quadruple

It is important that both Beltrami- dS/AdS spaces D_{\pm} share the $[C_O]$ with $Mink$ -space M as the tangent space at origin to both D_{\pm} so that together with the Poincaré double, dS/AdS double, \mathcal{D}_{\pm} , and dS/AdS doubles, there are also the $dS/AdS\text{-}\mathcal{P}$ doubles and the $dS/AdS\text{-}\mathcal{P}_2$ doubles $[\mathcal{D}_{\pm}, \mathcal{P}_2]_{D_{\pm}/M_{\pm}}$ with Einstein manifold M_{\pm} , respectively. Then, within $IM(1,3) \sim PGL(5,R)$ there are naturally six doubles and they form the quadruple \mathfrak{Q}_{PoR} with common Lorentz isotropy and $R_{\pm} = M \cap M_{\pm} \cap D_{+} \cap D_{-}$. But, as far as cosmological constants Λ_{\pm} are concerned, only two $dS/AdS\text{-}\mathcal{P}_2$ doubles $[\mathcal{D}_{\pm}, \mathcal{P}_2]_{D_{\pm}/M_{\pm}}$ are consistent kinematically in principle.

It should be emphasized that these different geometries are automatically combined together within LFT s (2.2) of the inertial motion group $IM(1,3) \sim PGL(5,R)$ based on the PoR_{cl} . And the homogeneous Lorentz group \mathcal{L} of common isotropy is just a subgroup of $IM(1,3)$ for all four relativistic kinematics.

It should also be noticed that these combined structures can also be reached even by contraction approach[7] under limit of $l \rightarrow 0$ or $\Lambda \rightarrow \infty$ with respect to a suitable contraction procedure based on the PoR_{cl} . In fact, for the cases of kinematics the contraction based on the PoR_{cl} more reasonable approach is to introduce a dimensionless contraction parameter ϵ and to replace l by ϵl , then taking the limit of $\epsilon \rightarrow 0$. It can be shown that under this contraction, the dS/AdS -groups and their invariant Beltrami- dS/AdS spaces D_{\pm} with common Lorentz group isotropy just contract to the \mathcal{P}_2 group and its invariant degenerate manifold M_{\pm} for R_{\pm} (3.5), which is the contraction form of domain conditions (4.4), respectively. While, under the contraction $\epsilon \rightarrow \infty$, both dS/AdS -invariant Beltrami- dS/AdS -spaces D_{\pm} become the \mathcal{P} -invariant $Mink$ -space M . Thus, even for the contraction approach[7], there are still the SR triple on three non-degenerate spacetime M/D_{\pm} and a relativistic quadruple in the sense of $\mathcal{L} = \mathcal{P} \cap \mathcal{P}_2 \cap \mathcal{D}_{+} \cap \mathcal{D}_{-}$ and $R_{\pm} = M \cap M_{\pm} \cap D_{+} \cap D_{-}$ ⁴.

In fact, this is also make sense physically for the inertial observers, whose world lines should always be time-like straightlines. No matter when and where they would change the inertial frames among four types of inertial coordinates with respect to four types of time and space coordinate translations in (1.1) of common Lorentz isotropy, they may change them. This is very important for their experiments and observations. Namely, among all relativistic kinematics with relevant inertial frames to be made up with respect to all possible time and space transformations and the required Lorentz isotropy, they should find that although Einstein's SR is perfect so far for the free space without gravity in ordinary scale, at cosmic scale our universe must kinematically prefer the dS SR with the $[\mathcal{D}_{+}, \mathcal{P}_2]_{D_{+}/M_{+}}$ of $\Lambda_{+} = 3l^{-2}$ and their Robertson-Walker-like counterparts.

⁴ In fact, the so-called infinite cosmological constant contraction and the dual Poincaré group had been studied in [24], but it didn't based on the PoR_{cl} , and relevant geometric results are not so meaningful. By means of the contraction, degenerate metrics (3.12) has also been found first in [25].

V. CONCLUDING REMARKS

It should be emphasized that all kinematics should be based on the PoR_{cl} and its symmetry. Then the inertial motion group follows for Newton's law of inertia, i.e. $IM(1, 3) \sim PGL(5, R)$. In addition to three kinds of Poincaré/ dS / AdS -invariant SR , with inertial motion the \mathcal{P}_2 kinematics can also be set up based on the PoR_{cl} on a degenerate Einstein manifold M_{\pm} with $\Lambda_{\pm} = \pm 3l^{-2}$ induced from C_O (3.4) as absolute for its space/time-like region R_{\pm} (3.5), respectively. With the Lorentz isotropy and the common R_{\pm} , there is the Poincaré double $[\mathcal{P}, \mathcal{P}_2]_{M/M_{\pm}}$ and dual to it there is also the dS double $[\mathcal{D}_+, \mathcal{D}_-]_{D_{\pm}}$. Together with other four doubles $[\mathcal{D}_{\pm}, \mathcal{P}]_{D_{\pm}/M}$ and $[\mathcal{D}_{\pm}, \mathcal{P}_2]_{D_{\pm}/M_{\pm}}$, there is the relativistic quadruple \mathfrak{Q}_{PoR} for three kinds of SR [10–14] as the SR triple on the intersected non-degenerate $Mink/dS/AdS$ -space[5], respectively. All these issues automatically appear in $IM(1, 3) \sim PGL(5, R)$.

For the fate of our universe, it is possibly a Robertson-Walker- dS with Λ_+ . So, the dS SR and associated double $[\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_+}$ should be paid much attention. Since the \mathcal{P}_2 -invariant M_+ is just for the space-like region R_+ of the Beltrami- dS space, it may not be so important at least at classical level. Whatever, the Robertson-Walker counterpart of the quadruple \mathfrak{Q}_{PoR} can be given. The dS SR with $[\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_+}$ and its Robertson-Walker version provide kinematics at the cosmic scale.

As was well-known, dynamics should coincide with kinematics. It seems reasonable to expect that these combinational structures may be shed light on dynamics of three kinds of special relativity and that of the relativistic quadruple.

It is worthy to mention that since the symmetry of the very special relativity (VSR)[27] is subgroup of \mathcal{P} , taking into account of four types of translations in (2.7) and (2.8) there should be the VSR of \mathcal{P}_2 and the quadruple \mathfrak{Q}_{VSR} for the VSR triple.

It should also be mentioned that for the maximal set of symmetries in 4d spacetime, like the $Mink$ spacetime, there is a 15-parameter conformal group $SO(2, 4)$. This is also true for the 4d dS/AdS spacetime, respectively (see, e.g. [26]). However, as was just mentioned this group contains always the special conformal transformations that are not LFT s and cannot preserve the inertial motion (2.1). It is still relevant to the group $IM(1, 3) \sim PGL(5, R)$ in the sense that there are 10-parameter kinematic subgroups in both them, although the whole conformal group is not a subset of the group $IM(1, 3)$. Of course, one may consider the conformal extension of $IM(1, 3) \sim PGL(5, R)$ that should contain the conformal extension of the relativistic quadruple \mathfrak{Q}_{PoR} .

It should be noticed first as far as some basic concepts and all principles are concerned, the approach to kinematics based on the PoR_{cl} and its symmetry, i.e. the LFT s of $IM(1, 3) \sim PGL(5, R)$, are completely different from that of Einstein's GR . The relation of our approach with GR is a very important issue. We will explore it elsewhere.

It should be noticed first as far as some basic concepts and all principles are concerned, the approach to kinematics based on the PoR_{cl} and its symmetry, i.e. the LFT s of $IM(1, 3) \sim PGL(5, R)$, are completely different from that of Einstein's GR . For example, there is no invariant metric for the whole LFT s of $IM(1, 3) \sim PGL(5, R)$, rather there are metric for kinematics of ten generators including either that of Lorentz isotropy or of space isotropy. While in GR , the pseudo-Riemann metric for spacetime is assumed to exist from beginning. In fact, from the beginning Einstein denied

the PoR and started from his equivalence principle and general covariance principle [4]. Although the $Mink$ space M , the Beltrami- dS/AdS space D_{\pm} of radius l , and the \mathcal{P}_2 -invariant degenerate manifold M_{\pm} all are Einstein manifolds of constant curvature, respectively, the cosmological constants correspond to each of them are different, except for the pairs of (D_+, M_+) and (D_-, M_-) . In addition, in view of GR , as long as the spacetime is curved, there should be no (global) inertial motions (2.1) so that it is hard to explain why there are (global) inertial motions (2.1) in the M_{\pm} and the D_{\pm} of constant curvature, respectively. On the other hand, although LFT s (2.2) may be viewed as a particular type of (differentiable) arbitrary coordinate transformations in view of Einstein's general covariance principle, it is hard to explain in GR why these LFT s form such a group $IM(1, 3) \sim PGL(5, R)$ that keeps inertial motions invariant, which completely departs from Einstein's original intention. In fact, there is no gravity for the group $IM(1, 3) \sim PGL(5, R)$ based on the PoR_{cl} and for all kinematics. How to introduce gravity in view of the PoR_{cl} , of course, is also very important. In order to describe gravity consistently with symmetry of the PoR in the localized version, it should be based on the local-globalization of the corresponding PoR with full kinematic symmetry of 1+3d spacetime (see, e.g. [14, 28]). Namely, to localize the kinematic symmetry first, then to connect the localized ones patch by patch globally with transition functions valued at the full symmetry.

We will study further these issues.

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